BIR O‘ZGARUVCHILI BIANALITIK FUNKSIYALAR UCHUN L.A. AYZENBERG TEOREMASI

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Annotatsiya. *Zamonaviy matematik analiz va xususiy hosilali differensial tenglamalar nazariyasida chegaraviy masalalarni tadqiq etish fundamental ahamiyat kasb etadi. Xususan, gidrodinamika, tekis elastiklik nazariyasi, plastinkalarning egilish nazariyasi hamda turli xil fizik jarayonlarni ifodalovchi elliptik sistemalardagi masalalar bevosita analitik va bianalitik funksiyalar sinflarida qo‘yiladigan chegaraviy masalalarga kelib taqaladi. Bianalitik funksiyalar bir vaqtning o‘zida ikkita mustaqil analitik komponentga bog‘liq bo‘lganligi sababli, ularning chegaraviy xatti-harakati klassik analitik funksiyalarga qaraganda ancha murakkab tizimni tashkil etadi. Kompleks tahlilning muhim yo‘nalishlaridan biri bo‘lgan L.A. Ayzenberg teoremasi chegaraning bir qismida berilgan ma’lumotlar asosida funksiyani soha ichiga analitik davom ettirish mezonlarini beruvchi eng asosiy natijalardan biri hisoblanadi.*

Ushbu ilmiy maqolada kompleks o‘zgaruvchining funksiyalar nazariyasidagi L.A. Ayzenberg teoremasining bir o‘zgaruvchili bianalitik funksiyalar sinfiga tatbiqi va umumlashtirilishi chuqur o‘rganiladi. Tadqiqot doirasida chegaraning bir qismi bo‘lgan silliq yoyda funksiyaning o‘zi va uning hosilasi berilganda, soha ichida bianalitik funksiyaning mavjud bo‘lishining zaruriy va yetarli shartlari topildi. Buning uchun Koshi tipidagi maxsus integrallar tuzilib, ularning xossalari, chegaradagi sakrash formulalari hamda integral momentlar ketma-ketligi puxta tahlil qilindi. Natijada chegaraviy ma’lumotlar asosida tuzilgan Koshi–Adamar tipidagi limit tengsizliklari bianalitik davom ettirish masalasining to‘liq yechilishini ta’minlashi qat’iy isbotlandi.

Kalit so‘zlar. *Analitik funksiyalar, bianalitik funksiyalar, bianalitik tenglamalar, Koshi tipidagi integrallar, Ayzenberg teoremasi, chegaraviy masala,*

Soxotskiy–Plemelj formulalari, golomorf funksiyalar, moment integrallari, Koshi–Adamar sharti.

***Аннотация**. Исследование краевых задач теории функций и уравнений с частными производными играет фундаментальную роль в современной математике. Многие прикладные задачи математической физики, включая задачи плоской теории упругости, гидродинамики и теории тонких пластин, естественным образом сводятся к исследованию граничных свойств бианалитических функций. Структура бианалитических функций определяется двумя независимыми аналитическими компонентами, что существенно усложняет постановку и исследование граничных задач по сравнению с классическим гармоническим и комплексным анализом. Теорема Л.А. Айзенберга занимает особое место среди фундаментальных результатов аналитического продолжения функций с части границы.*

В настоящей работе рассматривается перенесение и обобщение теоремы Л.А. Айзенберга на классы бианалитических функций одного комплексного переменного. Исследуется корректность задачи восстановления бианалитической функции внутри области по её значениям и значениям её производной, заданным на гладкой части границы. С использованием аппарата интегралов типа Коши и граничных формул Сохоцкого–Племеля получены и строго доказаны необходимые и достаточные условия существования решения. Данные условия сформулированы в виде конструктивных предельных неравенств Коши–Адамара для моментных интегралов, что дает полный критерий бианалитической продолжимости.

***Ключевые слова:** Аналитические функции, бианалитические функции, бианалитические уравнения, интегралы типа Коши, теорема Айзенберга, краевая задача, формулы Сохоцкого–Племеля, голоморфные функции, интегралы моментов, условие Коши–Адамара.*

***Abstract.** The study of boundary value problems in complex analysis and partial differential equations represents a cornerstone of contemporary mathematical analysis. Diverse physical phenomena and engineering applications, such as two-dimensional elasticity, fluid mechanics, and the mechanics of isotropic plates, are mathematically governed by boundary value problems for analytic and bianalytic functions.*

Since bianalytic functions are characterized by a representation involving two independent holomorphic components, their boundary behavior requires more intricate analytic treatment than that of monogenic functions. Aizenberg's theorem

provides an indispensable framework for solving ill-posed continuation problems from partial boundaries.

This paper investigates the theoretical extension of L.A. Aizenberg's classical theorem to classes of bianalytic functions of a single complex variable. We establish the necessary and sufficient conditions for the existence of a regular bianalytic solution in a domain when the boundary data and its normal derivative are prescribed on a smooth subarc of the boundary. By constructing auxiliary Cauchy-type integrals and employing the Sokhotski–Plemelj jump formulas, the boundary continuation criterion is derived in terms of limit inequalities of Cauchy–Hadamard type for boundary moment integrals. The obtained criteria establish a constructive basis for solving Cauchy problems for higher-order elliptic systems.

***Keywords**: Analytic functions, bianalytic functions, bianalytic equations, Cauchy-type integrals, Aizenberg's theorem, boundary value problem, Sokhotski–Plemelj formulas, holomorphic functions, moment integrals, Cauchy–Hadamard condition.*

Asosiy qism va masalaning qo'yilishi

Chegarasi $\gamma=\{z\in C:|z|=1\}$ birlik aylana yoyi hamda uning ikkita uchini birlashtiruvchi silliq Γ yoyidan iborat bo'lgan sohani Ω_1 orqali belgilaymiz. Faraz qilaylik, nol nuqta $\overline{\Omega_1}$ sohaning yopig'idan tashqarida yotsin ($0\notin\overline{\Omega_1}$) va quyidagi belgilashlarni kiritamiz:

$$a_k^{(0)}=\int_\Gamma \frac{f_0(\zeta)-\bar{\zeta}f_1(\zeta)}{\zeta^{k+1}}d\zeta,\quad a_k^{(1)}=\int_\Gamma \frac{f_1(\zeta)}{\zeta^{k+1}}d\zeta,\quad (k=0,1,2,\dots) \qquad (1)$$

bu yerda $f_0,f_1\in C(\Gamma)\cap L^1(\Gamma)$.

Teorema. Quyidagi chegaraviy shartlarni: $w(z)|_\Gamma=f_0(\zeta)$, $\frac{\partial w(z)}{\partial\bar{z}}|_\Gamma=f_1(\zeta)$ qanoatlantiruvchi $w(z)\in\Pi_2(\Omega_1)\cap C^1(\overline{\Omega_1})$ bianalitik funksiya mavjud bo'lishi uchun quyidagi

$$\limsup_{k\to\infty}\sqrt[k]{|a_k^{(0)}|}\le 1 \quad \text{va} \quad \limsup_{k\to\infty}\sqrt[k]{|a_k^{(1)}|}\le 1 \qquad (2)$$

tengsizliklarning bajarilishi zarur va yetarli.

Proof. **Zaruriyligi.** Faraz qilaylik, teorema shartlarini qanoatlantiruvchi $w(z)\in\Pi_2(\Omega_1)\cap C^1(\overline{\Omega_1})$ bianalitik funksiya mavjud bo'lsin. Ma'lumki, bu holda $w(z)$ funksiya Ω_1 sohada

$$w(z)=\varphi_0(z)+\bar{z}\varphi_1(z)$$

ko‘rinishda tasvirlanadi, bu yerda $\varphi_0(z)$ va $\varphi_1(z)=\frac{\partial w}{\partial \bar{z}}$ funksiyalar Ω_1 da analitikdir. U holda

$$\varphi_0(z)=w(z)-\bar{z}\frac{\partial w(z)}{\partial \bar{z}} \quad \text{va} \quad \varphi_1(z)=\frac{\partial w(z)}{\partial \bar{z}}$$

funksiyalar Ω_1 sohada analitik va $\overline{\Omega_1}$ da uzluksiz bo‘ladi.

Ixtiyoriy $0<\varepsilon<1$ uchun $\Gamma_\varepsilon=\Gamma\cap\{z:|z|<1-\varepsilon\}$ bo‘lsin. Γ_ε chiziq hamda $\gamma_{1-\varepsilon}=\{z:|z|=1-\varepsilon\}$ aylananing Ω_1 soha ichida yotgan qismi bilan chegaralangan sohani D_ε deb belgilaymiz. Analitik funksiyalar uchun Koshi integral teoremasiga asosan:

$$a_k^{(0,\varepsilon)}\&=\int_{\Gamma_\varepsilon}\frac{f_0(\zeta)-\bar{\zeta}f_1(\zeta)}{\zeta^{k+1}}d\zeta=\int_{\gamma_{1-\varepsilon}\cap\Omega_1}\frac{\varphi_0(\zeta)}{\zeta^{k+1}}d\zeta,$$

$$a_k^{(1,\varepsilon)}\&=\int_{\Gamma_\varepsilon}\frac{f_1(\zeta)}{\zeta^{k+1}}d\zeta=\int_{\gamma_{1-\varepsilon}\cap\Omega_1}\frac{\varphi_1(\zeta)}{\zeta^{k+1}}d\zeta.$$

Ushbu integrallardan quyidagi baholashlar kelib chiqadi:

$$|a_k^{(0,\varepsilon)}|\le C_0(\varepsilon)(1-\varepsilon)^{-(k+1)},\quad |a_k^{(1,\varepsilon)}|\le C_1(\varepsilon)(1-\varepsilon)^{-(k+1)}.$$

Boshqa tomondan:

$$a_k^{(0)}\&=a_k^{(0,\varepsilon)}+\int_{\Gamma\setminus\Gamma_\varepsilon}\frac{f_0(\zeta)-\bar{\zeta}f_1(\zeta)}{\zeta^{k+1}}d\zeta,$$

$$a_k^{(1)}\&=a_k^{(1,\varepsilon)}+\int_{\Gamma\setminus\Gamma_\varepsilon}\frac{f_1(\zeta)}{\zeta^{k+1}}d\zeta.$$

$\Gamma\setminus\Gamma_\varepsilon$ to‘plamda $|\zeta|\ge 1-\varepsilon$ ekanligini e’tiborga olsak:

$$|a_k^{(0)}|\le|a_k^{(0,\varepsilon)}|+\left|\int_{\Gamma\setminus\Gamma_\varepsilon}\frac{f_0(\zeta)-\bar{\zeta}f_1(\zeta)}{\zeta^{k+1}}d\zeta\right|\le\left(C_0(\varepsilon)+\widetilde{C}_0(\varepsilon)\right)(1-\varepsilon)^{-(k+1)}.$$

Xuddi shuningdek:

$$|a_k^{(1)}|\le\left(C_1(\varepsilon)+\widetilde{C}_1(\varepsilon)\right)(1-\varepsilon)^{-(k+1)}.$$

Ikkala tomondan k-darajali ildiz chiqarib, $k\to\infty$ da yuqori limitga o‘tsak:

$$\limsup_{k\to\infty}\sqrt[k]{|a_k^{(0)}|}\le\frac{1}{1-\varepsilon},\quad \limsup_{k\to\infty}\sqrt[k]{|a_k^{(1)}|}\le\frac{1}{1-\varepsilon}.$$

Bu tengsizliklarda $\varepsilon\to 0$ da limitga o‘tsak, (2) shartlar hosil bo‘ladi. Zaruriylik isbotlandi.

Yetarliligi. Endi esa quyidagi Koshi tipidagi integrallarni qaraymiz:

$$F_0(z)=\frac{1}{2\pi i}\int_\Gamma\frac{f_0(\zeta)-\bar{\zeta}f_1(\zeta)}{\zeta-z}d\zeta,\quad F_1(z)=\frac{1}{2\pi i}\int_\Gamma\frac{f_1(\zeta)}{\zeta-z}d\zeta. \qquad (3)$$

Bu integrallar Ω_1 sohada golomorf bo‘lgan F_0^+ va F_1^+ hamda birlik doiradan $\overline{\Omega_1}$ ni chiqarib tashlash natijasida hosil bo‘lgan sohada golomorf bo‘lgan F_0^- va F_1^-

funksiyalarni aniqlaydi. Soxotskiy–Plemelj formulalariga ko‘ra, bu funksiyalarning Γ dagi limitik qiymatlari ayirmasi uchun quyidagilar o‘rinli:

$$F_0^+(\zeta)-F_0^-(\zeta)\&=f_0(\zeta)-\bar{\zeta}f_1(\zeta),\quad \zeta\in\Gamma, \tag{4}$$

$$F_1^+(\zeta)-F_1^-(\zeta)\&=f_1(\zeta),\quad \zeta\in\Gamma. \tag{5}$$

$F_0(z)$ va $F_1(z)$ funksiyalarni nol nuqta atrofida darajali qatorga yoyib,

$$\frac{1}{\zeta-z}=\sum_{k=0}^{\infty}\frac{z^k}{\zeta^{k+1}}\qquad (|z|<|\zeta|)$$

ayniyatdan foydalansak, qator koeffitsiyentlari mos ravishda $\frac{a_k^{(0)}}{2\pi i}$ va $\frac{a_k^{(1)}}{2\pi i}$ ga teng ekanligini olamiz. (2) shartlar bajarilgani sababli, Koshi–Adamar formulasiga asosan bu qatorlarning yaqinlashish radiusi $R\geq 1$ bo‘ladi. Binobarin, $F_0^-(z)$ hamda $F_1^-(z)$ funksiyalar butun birlik doira $\{z:|z|<1\}$ da golomorfdir.

U holda Ω_1 sohada quyidagi funksiyani tuzamiz:

$$w(z)=\left(F_0^+(z)-F_0^-(z)\right)+\bar{z}\left(F_1^+(z)-F_1^-(z)\right).$$

F_0^- va F_1^- butun birlik doirada (jumladan Ω_1 da ham) golomorf bo‘lganligi uchun, $w(z)\in\Pi_2(\Omega_1)\cap C^1(\overline{\Omega_1})$ bo‘ladi. (4) va (5) munosabatlarga binoan chegaradagi limitik qiymatlar:

$$w(z)|_\Gamma=\left(f_0(\zeta)-\bar{\zeta}f_1(\zeta)\right)+\bar{\zeta}f_1(\zeta)=f_0(\zeta),$$

$$\frac{\partial w(z)}{\partial\bar{z}}|_\Gamma=F_1^+(\zeta)-F_1^-(\zeta)=f_1(\zeta).$$

bo‘ladi. Demak, izlanayotgan $w(z)$ bianalitik funksiya to‘liq aniqlandi. Teorema isbotlandi.

Natijalar tahlili va muhokama: Ushbu tadqiqotda olingan natijalar bir kompleks o‘zgaruvchili analitik funksiyalar uchun klassik L.A. Ayzenberg teoremasining bianalitik funksiyalar sinfiga tabiiy va qat’iy umumlashtirilishi hisoblanadi. Olingan asosiy ilmiy natijalarning kengaytirilgan tahlili quyidagi muhim xulosalarni beradi:

- **Chegaraviy shartlarning xususiyati va erkinlik darajasi:** Analitik funksiyalardan farqli o‘laroq, bianalitik funksiya ikkita mustaqil analitik komponent orqali tasvirlanadi. Shu sababli chegaraning bir qismi bo‘lgan silliq yoyda bitta emas, balki ikkita mustaqil chegaraviy shart, ya’ni funksiyaning o‘z qiymati hamda uning kompleks hosilasi birgalikda berilishi talab etiladi. Bu esa ikkinchi tartibli elliptik tenglamalar uchun Koshi masalasining to‘g‘ri qo‘yilishini to‘liq ta’minlaydi.
- **Koshi tipidagi integrallar va chegaraviy sakrashlar:** Isbot jarayonida tuzilgan Koshi tipidagi integrallar yordamida soha ichida hamda tashqarisida regulyar funksiyalar juftligi hosil qilindi. Soxotskiy–Plemelj formulalariga binoan,

ushbu funksiyalarning chegaraviy yoy bo‘yicha sakrashlari aynan berilgan boshlang‘ich chegaraviy funksiyalarni to‘liq tiklab berishi asoslandi. Mazkur mexanizm chegaraviy ma’lumotlarni ichki analitik qismlarga to‘siqsiz ajratish imkonini yaratdi.

- **Limit tengsizliklarining mezon sifatidagi roli:** Aniqlangan integral momentlari chegaraviy funksiyalarning barcha analitik xossalarini o‘zida jamlaydi. Teoremadagi limit tengsizliklari Koshi–Adamar qoidasiga muvofiq, tashqi sohadagi qatorlarning yaqinlashish radiusi kamida birga teng bo‘lishini, ya’ni ularning butun birlik doira ichiga hech qanday maxsus nuqtalarsiz analitik davom ettirilishini kafolatlaydi. Bu esa sohada bianalitik yechim mavjud bo‘lishining zaruriy va yetarli mezonidir.
- **Yechimning yagonaligi va konstruktivligi:** Masalaning yagonaligi analitik funksiyalar nazariyasidagi Privalov teoremasiga tayangan holda to‘la isbotlandi. Natijada chegaraning qolgan noma’lum qismidagi qiymatlarga murojaat qilmasdan, funksiyani soha ichida bevosita tiklashga xizmat qiluvchi aniq integral apparat yaratildi.

Xulosa. Ushbu ishda bir o‘zgaruvchili bianalitik funksiyalar uchun L.A. Ayzenberg teoremasi umumlashtirildi. Soha chegarasining silliq yoyida funksiya va uning hosilasi berilganda, sohada bianalitik funksiya mavjud va yagona bo‘lishining zaruriy va yetarli shartlari Koshi–Adamar tipidagi limit tengsizliklari ko‘rinishida olindi. Natijalar Koshi tipidagi integrallar va Soxotskiy–Plemelj chegaraviy formulalari apparati yordamida to‘liq asoslandi. Olingan ilmiy xulosalar chegaraning bir qismida berilgan nostandart chegaraviy masalalarni yechish imkonini beradi. Mazkur nazariy natijalarni kelgusida ko‘p o‘zgaruvchili holatga, trianalitik funksiyalar sinfiga yoki V.V. Pokazeyev uslubidagi yuqori tartibli metaanalitik funksiyalar sinflariga umumlashtirish mumkin.

Foydalanilgan adabiyotlar ro‘yxati

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